

ON THE ZEROS OF DIRICHLET L -FUNCTIONS. I

BY

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ABSTRACT. A mean value theorem for $\arg L(\frac{1}{2} + i(t+h), \chi) - \arg L(\frac{1}{2} + it, \chi)$ is established. This yields mean estimates for the number of zeros of $L(s, \chi)$ in small boxes.

1. **Statement of results.** As is customary we let $S(t, \chi) = \pi^{-1} \arg L(\frac{1}{2} + it, \chi)$, where the argument is determined by continuous variation on the half-line $\sigma + it$, $\sigma \geq \frac{1}{2}$. If t is the ordinate of a zero then we put

$$S(t, \chi) = \frac{1}{2}(S(t+0, \chi) + S(t-0, \chi)).$$

We assume throughout that χ is a primitive character mod q . The zeta function arises from the principal character with $q = 1$; thus $\zeta(s) = L(s, \chi_0)$, and we write $S(t) = S(t, \chi_0)$.

In 1944 A. Selberg [2] demonstrated that

$$\int_T^{T+H} S(t)^{2k} dt = c_k H(\log \log T)^k + O_k(H(\log \log T)^{k-1/2})$$

provided that $T^{1/2+a} \leq H \leq T$, $0 < a \leq \frac{1}{2}$, where

$$(1) \quad c_k = \frac{(2k)!}{(2\pi)^{2k} k!}.$$

We modify A. Selberg's approach to obtain our

Main Theorem. Let a_1, a_2 be fixed, $0 < a_i \leq \frac{1}{2}$ for $i = 1, 2$. Let χ be a primitive character (mod q), and suppose that $q \leq T^{1/4-a_1}$, $T^{1/2+a_2} \leq H \leq T$, and that $0 < h \leq (H - (H/\sqrt{T})^{1/8})$. Then

$$\int_T^{T+H} (S(t+h, \chi) - S(t, \chi))^{2k} dt = c_k H(2 \log(2 + h \log T))^k \cdot (1 + O((Ak)^{3k}(\log(2 + h \log T))^{-1/2})),$$

where $A > 0$ is a suitable absolute constant, and c_k is given by (1).

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Let $N(T, \chi)$ denote the number of zeros $\rho = \beta + iy$ of $L(s, \chi)$ with $0 < \beta < 1$, $0 \leq y \leq T$, zeros on the boundary being counted with weight one-half. It is well known (see [3]) that if $T \geq 0$ then

$$(2) \quad N(T, \chi) = \frac{T}{2\pi} \log \frac{qT}{2\pi} - \frac{T}{2\pi} - \frac{\chi(-1)}{8} + S(T, \chi) - S(0, \chi) + O\left(\frac{1}{T+1}\right).$$

Thus we easily derive the following

Corollary. *Under the above hypothesis,*

$$\begin{aligned} & \int_T^{T+H} (N(t+h, \chi) - N(t, \chi))^{2k} dt \\ &= H \left(\frac{h}{2\pi} \log qT \right)^{2k} \left(1 + O \left((Ak)^{4k} \frac{(\log(2+h \log T))^{1/2}}{2+h \log T} \right) \right) \\ & \qquad \qquad \qquad \text{unless } h \log T \rightarrow 0 \text{ as } T \rightarrow \infty, \\ &= O((Ak)^{4k} H) \text{ if } h \log T \rightarrow 0 \text{ as } T \rightarrow \infty \end{aligned}$$

where $A > 0$ is a suitable absolute constant.

We may remark here that implicit constants involved in O or $\ll$ in this paper do not depend on q , k and h unless it is written explicitly like O_k .

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2. Lemmas. The condition $q \leq T^{1/4-a_1}$ in the Main Theorem comes from

Lemma 1. *Let a_1, a_2 be fixed, $0 < a_i \leq 1/2$ for $i = 1, 2$. Let χ be a primitive character (mod q), and suppose that $q \leq T^{1/4-a_1}$, $T^{1/2+a_2} \leq H \leq T$. Then*

$$N_\chi(\sigma, T+H) - N_\chi(\sigma, T) = O(H(H/\sqrt{T})^{(1-2\sigma)/4} \log T)$$

uniformly for σ in $1/2 \leq \sigma \leq 1$, where $N_\chi(\sigma, T)$ denotes the number of zeros $\beta + iy$ of $L(s, \chi)$ with $\sigma < \beta$ and $0 < y < T$.

Proof. We will follow the argument of A. Selberg (cf. [2]) and we will state only the results in each step.

(i) We use A. F. Lavrik's approximate functional equation in the following form (cf. [1])

$$L(s, \chi) = \sum_{n \leq x} \frac{\chi(n)}{n^s} + \epsilon(\chi) \left(\frac{q}{\pi} \right)^{1/2-s} \frac{\Gamma(1/2(1-s+a))}{\Gamma(1/2(s+a))} \sum_{n \leq y} \frac{\overline{\chi(n)}}{n^{1-s}} + R_{xy},$$

where

$$s = \sigma + it, \quad 0 < \sigma < 1, \quad a = \frac{1}{2}(1 - \chi(-1)), \quad x = \Delta\sqrt{q|t|/2\pi}, \quad y = \frac{1}{\Delta}\sqrt{q|t|/2\pi},$$

$$\epsilon(\chi) = (-1)^a \frac{1}{\sqrt{q}} \sum_{n=1}^q \chi(n) e^{2\pi i(n/q)} \quad \text{and} \quad R_{xy} \ll \sqrt{q}(y^{-\sigma} + x^{\sigma-1}(qt')^{1/2-\sigma}) \log^2 |t|$$

with $t' = \max(|t|, 1)$.

(ii) Using (i) if $T > 10$, $\sqrt{T} \leq H \leq T$, μ_1 and μ_2 are positive coprime integers less than $T/10$, we have for $\frac{1}{2} < \sigma \leq 1$,

$$\begin{aligned} & \int_T^{T+H} |L(\sigma + it, \chi)|^2 \left(\frac{\mu_1}{\mu_2}\right)^{it} dt \\ &= \chi(\mu_1) \overline{\chi}(\mu_2) \int_T^{T+H} \{(\mu_1 \mu_2)^{-\sigma} \zeta(2\sigma) + (\mu_1 \mu_2)^{\sigma-1} r^{2-4\sigma} \zeta(2-2\sigma)\} dt \\ &+ O((qT)^{1-\sigma} \mu_1 \mu_2 \log^2(qT)) \\ &+ O\left(q^{1-\sigma} HT^{-2} \left(\left(\frac{\mu_1}{\mu_2}\right)^{1/2} + \left(\frac{\mu_2}{\mu_1}\right)^{1/2}\right) \log(qT) \cdot \log T\right), \end{aligned}$$

where $r = \sqrt{qt/2\pi}$.

(iii) For $T^{1/2+a_2} \leq H \leq T$, $0 < a_2 \leq \frac{1}{2}$, $\xi = (H/\sqrt{T})^{1/4}$, $\frac{1}{2} + 1/\log \xi \leq \sigma \leq 1$ and $q \leq T^{1/4-a_1}$,

$$\int_T^{T+H} |L(\sigma + it, \chi) \psi(\sigma + it, \chi)|^2 dt < H + O(H(H/\sqrt{T})^{(1-2\sigma)/4}),$$

where $\psi(s, \chi) = \sum_{\nu < \xi} (\lambda_\nu / \nu^s) \overline{\chi}(\nu)$ and λ_ν is defined as follows: for $\xi > 1$ and $\frac{1}{2} < \sigma \leq 1$ and $1 \leq \nu < \xi$

$$\lambda_\nu = \frac{\nu^{2\sigma}}{\sum_{\rho < \xi} \mu^2(\rho) / \phi_{2\sigma}(\rho)} \sum_{\rho < \xi/\nu} \frac{\mu(\rho\nu) \mu(\rho)}{\phi_{2\sigma}(\rho\nu)}$$

with $\phi_\gamma(\rho) = \rho^\gamma \sum_{d|\rho} \mu(d) d^{-\gamma}$ for a positive integer ρ and $\mu(d)$ is the Möbius function.

Proof. Same computation as A. Selberg (cf. [2]) leads to the following.

$$\begin{aligned} & \int_T^{T+H} |L(\sigma + it, \chi) \psi(\sigma + it, \chi)|^2 dt \\ &< H + O(H\xi^{1-2\sigma}) + O((qT)^{1-\sigma} \xi^{4-2\sigma} \log^2(qT) \cdot \log^2 T) \\ &+ O(q^{1-\sigma} HT^{-\sigma/2} \log(qT) \cdot \log^4 T) \\ &< H + O(H(H/\sqrt{T})^{(1-2\sigma)/4}) + O(q^{1-\sigma} HT^{1/2-\sigma-(2a_1-1)/8} \cdot \log^2 T \cdot \log^2(qT)) \\ &+ O(q^{1-\sigma} HT^{1/8-\sigma/2} \log^4 T \cdot \log(qT)). \end{aligned}$$

Hence if $q \leq T^{(1/4-\epsilon_1)}$ with $a_1 > 0$, the last two remainder terms become $O(H(H/\sqrt{T})^{(1-2\sigma)/4})$ and we get the conclusion.

(iv) Using Littlewood's lemma (cf. p. 106 in [2]) we get our conclusion.

Q.E.D.

Let $a(p)$ be a real or complex valued function of prime numbers p . For simplicity we write

$$F_a(x) = \sum_{p \leq x} \frac{|a(p)|^{2a}}{p^a}$$

for a real number a .

Lemma 2. Assume that $F_a(x) \ll 1$ for $a \geq 2$. Then

$$\sum_{p_i < x}^* \frac{a(p_1)a(p_2) \cdots a(p_k)\overline{a(p_{k+1})} \cdots \overline{a(p_{2k})}}{\sqrt{p_1 p_2 \cdots p_k p_{k+1} \cdots p_{2k}}} \\ = \begin{cases} k! F_1(x)^k + O(k! F_1(x)^{ov(k-2)}) & \text{if } F_1(x) \rightarrow \infty \text{ as } x \rightarrow \infty, \\ O(k!) & \text{if } F_1(x) \ll 1, \end{cases}$$

where in $\sum_{p_i < x}^*$ we sum over all primes $p_1, \dots, p_{2k}$ satisfying $p_1 \cdots p_k = p_{k+1} \cdots p_{2k}$ and $p_i < x$ for $i = 1, 2, \dots, 2k$, and $ov(k-2) = \max\{0, k-2\}$.

Proof. The expressions of k by the sum of n natural numbers can be ordered lexicographically when we identify two expressions up to permutations. Let $\sigma_k(n)$ be the number of such expressions. Let $\Phi(k-b, \nu_b)$ be the ν_b th expression $(c_1, c_2, \dots, c_{k-b})$ of k by the sum of $k-b$ natural numbers; $k = c_1 + c_2 + \cdots + c_{k-b}$. Let

$$A_{k-b, \nu_b} = \sum_{p_i < x}' \frac{|a(p_1)|^{2c_1} |a(p_2)|^{2c_2} \cdots |a(p_{k-b})|^{2c_{k-b}}}{p_1^{c_1} \cdots p_{k-b}^{c_{k-b}}}$$

for $\nu_b = 1, 2, \dots, \sigma_k(k-b)$, where in $\sum_{p_i < x}'$ we sum over all $p_1, \dots, p_{k-b}$ with $p_i < x$ for $i = 1, \dots, k-b$ whenever $p_1, \dots, p_{k-b}$ are all different, and $(c_1, c_2, \dots, c_{k-b}) = \Phi(k-b, \nu_b)$. Also let

$$\bar{A}_{k-b, \nu_b} = F_{c_1}(x) F_{c_2}(x) \cdots F_{c_{k-b}}(x) \\ = \left(\sum_{p < x} \frac{|a(p)|^{2c_1}}{p^{c_1}} \right) \cdots \left(\sum_{p < x} \frac{|a(p)|^{2c_{k-b}}}{p^{c_{k-b}}} \right).$$

Now

$$(1) \quad \sum_{p_i < x}^* \frac{a(p_1)a(p_2) \cdots a(p_k)\overline{a(p_{k+1})} \cdots \overline{a(p_{2k})}}{\sqrt{p_1 p_2 \cdots p_{2k}}} = \sum_{p_i < x}^* \frac{|a(p_1)|^2 \cdots |a(p_k)|^2}{p_1 \cdots p_k}$$

$$= \sum_{b=0}^{k-1} \sum_{\nu_b=1}^{\sigma_k(k-b)} (k-b)! A_{k-b, \nu_b}.$$

We remark that $\bar{A}_{k-b, \nu_b} = A_{k-b, \nu_b} + \sum_{b'=b+1}^{k-1} \sum_{\nu_{b'}=1}^{\sigma_k(k-b')} A_{k-b', \nu_{b'}}$. Substituting this into the above, we get

$$(1) = k! F_1(x)^k + \sum_{b=1}^{k-1} (b-k) \cdot (k-b)! \sum_{\nu_b=1}^{\sigma_k(k-b)} \bar{A}_{k-b, \nu_b}$$

since $\bar{A}_{k, \nu_0} = (\sum_{p < x} |a(p)|^2/p)^k = F_1(x)^k$. Now by the assumption if $F_1(x) \rightarrow \infty$ as $x \rightarrow \infty$

$$(2) \quad \bar{A}_{k-b, \nu_b} \ll F_1(x)^{O\nu(k-2)} \quad \text{for } 1 \leq b \leq k-1.$$

Hence it is enough to show that

$$(3) \quad (b-k) \cdot (k-b)! \ll \sigma_k(k-b) \ll (k-1)!$$

for $b = 1, 2, \dots, k-1$. Moreover it is enough to estimate this for sufficiently big k . We define $\sigma'_k(n)$ by

$$\frac{x^n}{(1-x)^n} = \sum_{k=n}^{\infty} \sigma'_k(n) x^k;$$

then $\sigma_k(n) \leq \sigma'_k(n)$. Since

$$\frac{x^n}{(1-x)^n} = \sum_{k \geq n} \frac{(k-1)!}{(n-1)!(k-n)!} x^k,$$

$\sigma_k(n) \leq \sigma'_k(n) = (k-1)!/(n-1)!(k-n)!$. Now (3) is clear for $b \ll 1$ or $k-b \ll 1$.

In other cases since $(k-b)^2 \leq b!$ for sufficiently big k ,

$$|(b-k) \cdot (k-b)! \sigma_k(k-b)| \leq \frac{(k-1)!}{(k-b-1)!(k-k+b)!} (k-b)! \cdot (k-b)$$

$$\leq (k-1)!(k-b)^2/b! \leq (k-1)!.$$

Hence by (2) and (3) $(1) = k! F_1(x)^k + O(k! F_1(x)^{O\nu(k-2)})$. If $F_1(x) \ll 1$, $\bar{A}_{k-b, \nu_b} \ll 1$ for $0 \leq b \leq k-1$. Hence we get our conclusion. Q.E.D.

Lemma 3. Assume that $F_\alpha(x) \ll 1$ for $\alpha \geq 2$ and $F_{1/2}(x) \ll x^c$ with some positive c . Then for $x = T^{(a_2 + 1/2)/2k(c+1)}$

$$\int_T^{T+H} \left| \operatorname{Im} \sum_{p < x} \frac{a(p)}{p^{1/2+it}} \right|^{2k} dt = \begin{cases} \frac{2k!}{2^{2k} k!} HF_1(x)^k + O\left(\frac{2k!}{2^{2k} k!} HF_1(x)^{O\nu(k-2)}\right) & \text{if } F_1(x) \rightarrow \infty \text{ as } x \rightarrow \infty, \\ O(k^k H) & \text{if } F_1(x) \ll 1, \\ O(k^k A^{2k} H) & \text{always,} \end{cases}$$

where A satisfies $|a(p)| \leq A \log p / \log x$ for any $p \leq x$.

Proof (First case). We write

$$\tau = \sum_{p < x} \frac{a(p)}{p^{1/2+it}},$$

and

$$\operatorname{Im} \sum_{p < x} \frac{a(p)}{p^{1/2+it}} = \frac{\tau - \bar{\tau}}{2i}$$

and

$$\begin{aligned} \left| \operatorname{Im} \sum_{p < x} \frac{a(p)}{p^{1/2+it}} \right|^{2k} &= \frac{1}{2^{2k}} \sum_{b=0}^{2k} (-1)^{k-b} \binom{2k}{b} \cdot \int_T^{T+H} \tau^b \bar{\tau}^{2k-b} dt \\ &= \frac{1}{2^{2k}} \sum_{b=0}^{2k} (-1)^{k-b} \binom{2k}{b} \sum_{p < x} \frac{a(p_1) \cdots a(p_b) \overline{a(p_{b+1})} \cdots \overline{a(p_{2k})}}{\sqrt{p_1 p_2 \cdots p_{2k}}} \int_T^{T+H} \left(\frac{p_{b+1} \cdots p_{2k}}{p_1 \cdots p_b} \right)^{it} dt. \end{aligned}$$

If $p_{b+1} \cdots p_{2k} \neq p_1 \cdots p_b$,

$$\int_T^{T+H} \left(\frac{p_{b+1} \cdots p_{2k}}{p_1 \cdots p_b} \right)^{it} dt = O\left(\frac{1}{\left| \log \frac{p_{b+1} \cdots p_{2k}}{p_1 \cdots p_b} \right|} \right) = O(x^{2k}).$$

Hence for $b \neq k$,

$$\int_T^{T+H} \tau^b \bar{\tau}^{2k-b} dt = O(x^{2k} F_{1/2}(x)^{2k}) = O(x^{2k(1+c)}) = O(H).$$

If $b = k$, we get by Lemma 1

$$\begin{aligned} \int_T^{T+H} |\tau|^{2k} dt &= H \sum_{p_i < x}^* \frac{a(p_1) \cdots a(p_k) \overline{a(p_{k+1})} \cdots \overline{a(p_{2k})}}{p_1 \cdots p_k} + O(H) \\ &= k! HF_1(x)^k + O(Hk! F(x)^{O\nu(k-2)}), \end{aligned}$$

where the meaning of $\sum_{p_i < x}^*$ is the same as in Lemma 1. Hence

$$\begin{aligned} \int_T^{T+H} \left| \operatorname{Im} \sum_{p < x} \frac{a(p)}{p^{1/2+it}} \right|^{2k} dt &= \frac{1}{2^{2k}} \binom{2k}{k} H k! F_1(x)^k \\ &\quad + O\left(\frac{1}{2^{2k}} \binom{2k}{k} H k! F_1(x)^{O(\nu(k-2))}\right) \\ &\quad + O\left(\frac{1}{2^{2k}} \sum_{b=0; b \neq k}^{2k} \binom{2k}{b} H\right) \\ &= \frac{2k!}{2^{2k} k!} H F_1(x) + O\left(\frac{2k!}{2^{2k} k!} H F_1(x)^{O(\nu(k-2))}\right). \end{aligned}$$

(Other cases). For $b \neq k$, $\int_T^{T+H} r^{b-2k-b} dt = O(H)$ as above. For $b = k$,

$$\int_T^{T+H} |r|^{2k} dt = H \sum_{p_i < x}^* \frac{a(p_1) \cdots a(p_k) \overline{a(p_{k+1})} \cdots \overline{a(p_{2k})}}{p_1 \cdots p_k} + O(H) = O(k!H)$$

by Lemma 1. Hence we get the second part. The third part comes from Lemma 4 below. Q.E.D.

For $x \geq 2$ we define (see p. 47 of [3]) a number $\sigma_x(t)$ by

$$\sigma_x(t) \equiv \sigma_x(x, t) = \frac{1}{2} + 2 \max_{\rho} (\beta - \frac{1}{2}, 2/\log x),$$

where $\rho = \beta + i\gamma$ runs through the zeros of $L(s, \chi)$ for which $|t - \gamma| \leq x^{3(\beta-1/2)/\log x}$. Now modifying A. Selberg's arguments (i.e., Lemma 12 on p. 121 of [2] and Lemma 13 on p. 123 of [2]) we get the following two lemmas. We omit these proofs. We use Lemma 1 to prove Lemma 4. The condition on h in the Main Theorem comes from Lemma 4.

Lemma 4. Under the same hypothesis as Lemma 1, for ξ satisfying $1 \leq \xi \leq x^{8k}$ and $x^3 \xi^2 \leq (H/\sqrt{T})^{1/4}$, for ν in $0 \leq \nu \leq 8k$, and for $h = h(T) \leq H - (H/\sqrt{T})^{1/8}$,

$$\int_T^{T+H} (\sigma_x(t+h) - \frac{1}{2})^\nu \xi^{\sigma_x(t+h)-1/2} dt \ll \frac{H A_1^k \nu!}{(\log x)^\nu},$$

where A_1 is some positive absolute constant.

Lemma 5. Let $H > 1$, $1 < y \leq H^{1/k}$ and $|a(p)| < A \log p / \log y$ for $p < y$ and $|a'(p)| \leq B$ for $p < y$. Then

$$\int_0^H \left| \sum_{p < y} \frac{a(p)}{p^{1/2+it}} \right|^{2k} dt = O((A^2 k)^k A_1^k H)$$

and

$$\int_0^H \left| \sum_{p < y} \frac{a'(p)}{p^{1+2it}} \right|^{2k} dt = O((B^2 k)^k A_1^k H),$$

where A_1 's are some positive absolute constants.

3. Proof of theorem. We need the following expression for $x \geq 2$ due to A. Selberg (cf. [3]):

$$S(t, \chi) - \frac{1}{\pi} \operatorname{Im} \sum_{p < x^3} \frac{\chi(p)}{p^{1/2+it}} = \sum_{i=1}^5 O(B_i(t)),$$

where

$$\begin{aligned} B_1(t) &= \left| \sum_{p < x^3} \frac{\Lambda(p) - \Lambda_x(p)}{p^{1/2+it} \log p} \cdot \chi(p) \right|, & B_2(t) &= (\sigma_x(t) - 1/2) \log x, \\ B_3(t) &= (\sigma_x(t) - 1/2) \log(q(1 + |t|)), & B_4(t) &= \left| \sum_{p < x^{3/2}} \frac{\Lambda_x(p^2) \chi(p^2)}{p^{1+2it} \log p} \right|, \\ B_5(t) &= (\sigma_x(t) - 1/2) x^{\sigma_x(t)-1/2} \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p < x^3} \frac{\Lambda_x(p) \log(xp)}{p^{\sigma+it}} \chi(p) \right| d\sigma \end{aligned}$$

and

$$\begin{aligned} \Lambda_x(n) &= \Lambda(n) && \text{if } 1 \leq n \leq x, \\ &= \Lambda(n) \frac{(\log x^3/n)^2 - 2(\log x^2/n)^2}{2(\log x)^2} && \text{if } x \leq n \leq x^2, \\ &= \Lambda(n) \frac{(\log x^3/n)^2}{2(\log x)^2} && \text{if } x^2 \leq n \leq x^3. \end{aligned}$$

For simplicity we write

$$\|f\| = \left(\int_T^{T+H} |f(t)|^{2k} dt \right)^{1/2k}.$$

We use the above expression for $x = T^{a_2/60k}$.

Now

$$A_{x^3}(t) \stackrel{\text{def}}{=} S(t+h, \chi) - S(t, \chi) - f_{x^3}(t) = \sum_{i=1}^5 O(B_i(t+h)) + \sum_{i=1}^5 O(B_i(t)),$$

where $f_3(t) = \pi^{-1} \operatorname{Im} \sum_{p < x} a(p)/p^{(1/2+it)}$ with $a(p) = \chi(p)(e^{-ih \log p} - 1)$. By Lemma 5 $\|B_i(t+h)\| \ll k^{1/2} H^{1/2k}$ and $\|B_i(t)\| \ll k^{1/2} H^{1/2k}$ for $i = 1$ and 4. By Lemma 4 with $\xi = 1$ and by the assumption on q , $\|B_i(t+h)\| \ll k^{1/2} H^{1/2k}$ and $\|B_i(t)\| \ll k^2 H^{1/2k}$ for $i = 2$ and 3. By Lemmas 4 and 5 as on p. 126 of [2],

$$\|B_5(t+h)\| \ll k^{3/2} H^{1/2k} \quad \text{and} \quad \|B_5(t)\| \ll k^{3/2} H^{1/2k}.$$

Hence for $x = T^{a_2/60k}$, $\|A_{x_3}(t)\| \ll k^2 H^{1/2k}$. For $T^{a_2/20k} \leq x \leq H^{1/k}$,

$$\begin{aligned} \|A_x(t)\| &\leq \|A_{T^{a_2/20k}}(t)\| + \left\| \sum_{T^{a_2/20k} \leq p \leq x} \frac{a(p)}{p^{1/2+it}} \right\| \\ &\ll k^2 H^{1/2k} + k^{1/2} H^{1/2k} \ll k^2 H^{1/2k} \quad \text{by Lemma 5.} \end{aligned}$$

Hereafter we take $x = T^{(a_2 + 1/2)/2k(c+1)}$, where c is some nonnegative constant satisfying $F_{1/2}(x) = \sum_{p < x} a(p)/\sqrt{p} \ll x^c$.

Now in using Lemma 3 in our case

$$\begin{aligned} F_1(x) &= \sum_{p < x} \frac{|e^{-ih \log p} - 1|^2}{p} = 2 \sum_{p < x} \frac{1 - \cos(h \log p)}{p} \\ &= 2 \sum_{p < x} \frac{1 - \cos(h \log p)}{\log p} \frac{\log p}{p} = 2 \int_1^x \frac{1 - \cos(h \log \xi)}{\log \xi} d(\log \xi + O(1)) \end{aligned}$$

since $\sum_{p < \xi} \log p/p = \log \xi + O(1)$. Hence

$$F_1(x) = 2 \int_0^{h \log x} \frac{1 - \cos u}{u} du + O(1) = 2 \log(h \log x) + O(1).$$

Hence if $h \log T \rightarrow \infty$ as $T \rightarrow \infty$, then $F_1(x) \rightarrow \infty$ and by Lemma 3

$$\|f_x(t)\|^{2k} = c(k) H F_1(x)^k + O(c(k) H F_1(x)^{o\nu(k-2)}).$$

Hence

$$\begin{aligned} \|S(t+h, \chi) - S(t, \chi)\|^{2k} &= \|A_x(t) + f_x(t)\|^{2k} \\ &= c(k) H (2 \log(h \log T))^k + O((Ak)^{4k} H (\log(h \log T))^{k-1/2}) \end{aligned}$$

if $h \log T \rightarrow \infty$ as $T \rightarrow \infty$.

If $h \log T \ll 1$, using Lemma 3 again we get similarly

$$\|S(t+h, \chi) - S(t, \chi)\|^{2k} \ll (Ak)^{4k} H.$$

Hence we get our conclusion. Q.E.D.

4. Concluding remarks. We can get mean estimates of $\sum_{m=0}^L \sum_{\chi} b(m)g(\chi)S(t+bm, \chi)$ similarly as we did in §3, where $b(0), \dots, b(L)$ are real numbers and $g(\chi)$ is a complex number for a character $\chi \bmod q$. Instead of describing a general theorem to this, we will give only some examples of this. We always assume that $q \leq T^{1/4-a_1}$ and $T^{1/2+a_2} \leq H \leq T$ for a fixed a_1 and a_2 in $0 < a_i \leq 1/2$ for $i = 1, 2$.

$$(i) \quad \int_T^{T+H} \left(\sum'_{\chi} S(t, \chi) \right)^{2k} dt = c(k)H(\phi(q) - 1)^{2k}(\log \log T)^k \\ + O_k(H(\phi(q) - 1)^{2k-1}(\log \log T)^{k-1/2}),$$

where in $\sum'_{\chi} \chi$ runs over all nonprincipal characters mod q .

$$(ii) \quad \int_T^{T+H} \left(\sum'_{\chi} S(t+h, \chi) - \sum'_{\chi} S(t, \chi) \right)^{2k} dt \\ = c(k)(\phi(q) - 1)^{2k}H(2 \log(2+h \log T))^k \\ \cdot (1 + O((Ak)^{3k}(\phi(q) - 1)^{-1}(\log(2+h \log T))^{-1/2}))$$

uniformly for h satisfying $0 < h \leq (H - (H/\sqrt{T})^{1/8})$, where A is some positive absolute constant.

$$(iii) \quad \int_T^{T+H} \left(\frac{S(t, \chi) + S(t+h, \chi)}{2} \right)^{2k} dt = c_k H(\log \log T)^k + O_k(H(\log \log T)^{k-1/2})$$

if $h \log T \rightarrow 0$ as $T \rightarrow \infty$.

(iv) For any Δ satisfying $\Delta \log T \rightarrow 0$ as $T \rightarrow \infty$, there exists a function $\Phi(T)$ satisfying $\Phi(T) \rightarrow \infty$ as $T \rightarrow \infty$ such that

$$\int_T^{T+H} \left(\frac{1}{\Delta} \int_t^{t+\Delta} S(u, \chi) du \right)^{2k} dt = c_k H(\log \log T)^k + O_k(H(\log \log T)^k / \Phi(T)).$$

Also

$$\int_T^{T+H} \left(\frac{1}{\Delta(\phi(q) - 1)} \sum'_{\chi} \int_t^{t+\Delta} S(u, \chi) du \right)^{2k} dt \\ = c_k H(\log \log T)^k + O_k(H(\log \log T)^k / \Phi(T)).$$

$$(v) \quad \int_T^{T+H} S(t, \chi)^{2k} dt = c(k)H(\log \log T)^k + O_k(H(\log \log T)^{k-1/2}).$$

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